Kazhdan constants of group extensions

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July 27, 2010

Abstract

We give bounds on Kazhdan constants of abelian extensions of (finite) groups. As a corollary, we improved known results of Kazhdan constants for some meta-abelian groups and for the relatively free group in the variety of p-groups of lower p-series of class 2. Furthermore, we calculate Kazhdan constants of the tame automorphism groups of the free nilpotent groups.

1 Introduction.

Property (T) was introduced by Kazhdan [Kaz] in 1967. It founds numerous applications in various areas of mathematics, in particular for estimating quantitative behavior of the mixing time of random walks and the expansion of the Cayley graphs of finite groups. For introduction to the subject and applications we refer the reader to [BHV, Lub].

Definition 1.1. Let Γ be a discrete group, $S \subset \Gamma$ a subset, $\epsilon > 0$, and let $(\rho, \mathcal{H}_\rho)$ be a unitary representation of Γ. A vector $0 \neq v \in \mathcal{H}_\rho$ is called (S, ϵ)-invariant, if for every $s \in S$, we have $\|\rho(s)v - v\| \leq \epsilon\|v\|$.

Definition 1.2. Let Γ be a discrete group and let S be a finite generating subset.

1. The group Γ is said to have *Kazhdan property (T)* if there exists an $\epsilon > 0$ such that every unitary representation with (S, ϵ)-invariant vector, contains a non-zero Γ-invariant vector. In that case ϵ is called a *Kazhdan constant* of Γ with respect to S.

2. More generally, if $\mathcal{C}$ is a class of unitary representations of Γ, we say that ϵ is a *Kazhdan constant* of Γ, with respect to the set S and relative to the class $\mathcal{C}$, if every representation in $\mathcal{C}$ with (S, ϵ)-invariant vector, contains a non-zero Γ-invariant vector.

It is a standard observation that a group Γ has T iff there exists a finite generating multiset S (i.e. an object like a set except that multiplicity is significant) of Γ and an $\epsilon > 0$ such that

$$\inf_{\rho \in \Gamma_0^*} \inf_{0 \neq v \in \mathcal{H}_\rho} \frac{1}{|S|} \sum_{s \in S} \frac{\|\rho(s)v - v\|^2}{\|v\|^2} \geq \epsilon,$$

where Γ_0^* denote the space of unitary representations of Γ with no invariant non-zero vectors.

Definition 1.3. The number ϵ above is called a *average Kazhdan constant* of Γ with respect to S.

This definition is equivalent to the previous definition, they only differ in ϵ. One can check that the infimum in the average Kazhdan constant is obtained at an irreducible representation. In general, for a group Γ with a Kazhdan constant ϵ_1 with respect to S, and with an average Kazhdan constant ϵ_2 (with respect to S), we have the following inequalities $\epsilon_1^2 \geq \epsilon_2 \geq \frac{\epsilon_1^2}{|S|}$.

It turns out (see subsection 1.2) that in case of a finite group, it is more appropriate to define a Kazhdan constant of a group with respect to the mean instead of the maximum over the generators. So throughout this paper the notation *Kazhdan constant* will be used for the case of infinite groups and an *average Kazhdan constant* for finite groups.

Let Γ be a finitely generated group with a Kazhdan constant ϵ_1 with respect to S_1. Let S_2 be another generating set then it is not hard to prove (see [Lub], rem. 3.2.5) that there exists $\epsilon_2 = \epsilon_2(\epsilon_1, S_1, S_2)$ such that the Kazhdan constant of Γ with respect to S_2 is greater than or equal to ϵ_2. For the interesting question of the dependance of Kazhdan constants on the generating sets, see [ALW, GZ, Kas1]. This bring us to the following definition:

Definition 1.4. Let Γ be a finite group. Given $0 < \epsilon < 1$, we define $g_\epsilon(\Gamma)$ as the minimal integer N, such that there is a generating set S of Γ, with $|S| \leq N$, and the average Kazhdan constant of Γ with respect to S is greater than or equal to ϵ.

The number $g_\epsilon(\Gamma)$ is important in many theoretical and practical computational problems, hence we ask the following natural question.

Question 1.5. For a finite group Γ and $0 < \epsilon < 1$, what is $g_\epsilon(\Gamma)$?

N. Alon and Y. Roichman (Corollary 1 in [AR], see also [LR, LS]) showed that $g_\epsilon(\Gamma) \leq O_\epsilon(\log |\Gamma|)$ for any $0 < \epsilon < 1$. This bound is sharp for the case of abelian groups. However, for any finite non-abelian simple group Γ (except Suzuki), it has been proved recently by Kassabov, Lubotzky and Nikolov [KLN], that $g_\epsilon(\Gamma) = O(1)$ for some $0 < \epsilon < 1$.

One of the goals of this paper is to present a method for estimating $g_\epsilon(\Gamma)$ for some non-simple groups.

1.1 Main Results

Theorem 1.6. *Let A, H be two finite groups with A abelian, such that H acts on A. Let Γ be an extension of A by H. Let $S \subset H$, $\tilde{S}$ a lifting of the set S to Γ, $B \subset A$ a subset and $\tilde{B} = \{a^x \mid a \in B, x \in H\}$ be the union of the orbits. Assume that the following hold:*

1. *H is generated by S and has average Kazhdan constants ϵ_H with respect to S.*

2. *A is generated by $\tilde{B}$ and has average Kazhdan constants ϵ_A with respect to $\tilde{B}$.*

Then the group Γ has average Kazhdan constants

$$\epsilon > \frac{\epsilon_H \epsilon_A}{512 \cdot \left(1 + \frac{|S|}{|B|} + \frac{|B|}{|S|}\right)}$$

with respect to the set of generators $\tilde{S} \cup B$.

Note that the conclusion of the theorem holds for an arbitrary lifting $\tilde{S}$ of S, with the same constant. Moreover, the assumptions on A are with respect to the generating set $\tilde{B}$ which may be very large (of size $|B| \cdot |H|$) whereas the generating set of Γ might be much smaller.

This theorem is an extension of a work by Alon, Lubotzky and Wigderson [ALW] who proved a similar result for the case of sem-direct product. In fact they showed that the semi-direct product is a group theoretic analog to the zig-zag product of Reingold, Vadhan and Wigderson [RVW]. Here we show that a similar result holds also for other cases of extensions as well.

The main idea of the proof of Theorem 1.6 is the connection between almost Γ-invariant vector and almost H-invariant measure on the unitary dual of A. If there exists an almost invariant vector with respect to the generating set of Γ, then one can associate with it a measure on the unitary dual of the abelian group A. We show that this measure is almost H-invariant under the dual action of H on $\hat{A}$, and this brings to a contradiction.

Our method is generic and makes no assumption on the finite groups A and H. However, for some specific groups (even infinite) A and H, one can do better. For example, see [Bur, Kas1, Sh1], where the authors study the group $EL_n(R) \ltimes R^n$ (where R is an associative ring) with respect to some unipotent matrices as a generating set.

The following is deduced from the main Theorem:

Corollary 1.7. *Let $n \in \mathbb{N}$, p a prime, and let C_p, C_{p^n} be the cyclic groups of order p and p^n respectively. Let Γ be any extension of the group algebra $\mathbb{F}_p[C_p]$ by C_{p^n}, where C_{p^n} acts by left translation (mod p) on $\mathbb{F}_p[C_p]$. Then there exists $\epsilon > 0$, such that $g_\epsilon(\Gamma) \leq O_\epsilon(n \log p)$.*

Note that by Alon-Roichman Theorem, $g_\epsilon(\Gamma) \leq O_\epsilon((n + p) \log p)$.
Theorem 1.6 covers the case of non-trivial extensions. For the case of central extension the following is a quantitative version of a result of Serre.

Theorem 1.8 (Serre). *Let Γ be a central extension of A by H, i.e $A \subseteq \mathbf{Z}(\Gamma)$. Let $\tilde{S} \subset \Gamma$ be a set of generators of Γ and let S be the projection of $\tilde{S}$ to H. Assume that ϵ_1 (resp. ϵ_2) is a Kaz. constant (resp. an average Kaz. constant) of H with respect to S. Then $\frac{\epsilon_1}{2 \cdot |S|}$ (resp. $\frac{\epsilon_2}{4}$) is a Kaz. constant (resp. an average Kaz. constant) of Γ with respect to $\tilde{S}$ with respect to any unitary representation of Γ which does not contains a one dimensional subrepresentation.*

For the proof we refer the reader to [HV] (pages 26-28 or Lemma 1.7.10 in [BHV]). Although the quantitative result is not stated there explicitly, it follows immediately from the proof. In subsection 3.4 we use this result for estimating Kazhdan constants of the tame automorphism groups of the free nilpotent groups, extending a result of Mallahi [MAK]. These estimations have applications for evaluating the mixing time of the product replacement algorithm.

To illustrate the use of the above result for concrete groups, we prove the following:
For a fixed prime p, let $\Gamma_{k,2}$ be be the relatively free group on k generators in the variety of p-groups of lower p-series of class 2; the precise definition is provided in Sec. 3.3.

Corollary 1.9. *For all $0 < \epsilon < \frac{1}{4}$ we have*

$$g_\epsilon(\Gamma_{k,2}) \le O_\epsilon(\sqrt{\log |\Gamma_{k,2}|}).$$

So our method improves Alon-Roichman result which requires $O_\epsilon(\log |\Gamma_{k,2}|)$ elements.

1.2   The Spectral Gap

Let $G = Cay(\Gamma, S)$ be the Cayley graph of a finite group Γ with respect to a symmetric generating set S ($S = S^{-1}$). Let A_r be the normalized adjacency matrix of the Cayley graph G. We denote by $\lambda(\Gamma, S)$ the second largest eigenvalue of A_r and we also define its *spectral gap* $\beta = 1 - \lambda(\Gamma, S)$ (for applications of β see [HLW]). It turns out that there is a tight connection between the average Kazhdan constant and the spectral gap of G, indeed we have (see fact 1.1 in [MW] and section 11 in [HLW]):

$$\beta = \inf_{\rho \in \Gamma_0^*} \inf_{0 \ne v \in \mathcal{H}_\rho} \frac{1}{2|S|} \sum_{s \in S} \frac{\|\rho(s)v - v\|^2}{\|v\|^2} = \inf_{\rho \in \Gamma_0^*} \inf_{0 \ne v \in \mathcal{H}_\rho} \frac{\|\frac{1}{|S|} \sum_{s \in S} \rho(s)v - v\|}{\|v\|},$$

where Γ_0^* denotes the space of unitary representations of Γ with no invariant non-zero vectors.

Let us now describe the structure of this paper. Section 2 is devoted to a proof of Theorem 1.6. In subsection 3.1 we show that if $n > 1$, then the cohomology group classifying the group extensions of $F_p[C_p]$ by C_{p^n}, is not trivial. Moreover, we prove the existence of "good orbits" in $F_p[C_p]$ under the action of C_{p^n}, and as a result we obtain Corollary 1.7. In subsection 3.3, we prove Corollary 1.9, and in subsection 3.4 we study the structure of the group of tame automorphisms of the free nilpotent group and calculate explicitly its Kazhdan constants.

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2 Proof of Theorem 1.6

Let $(\rho, \mathcal{H})$ be a non-trivial unitary irreducible representation of Γ and let

$$W_0 = \{w \in \mathcal{H} \mid \rho(a)w = w \quad \forall a \in A\}$$

be the space of A-invariant vectors. Since $A \lhd \Gamma$, W_0 is Γ invariant. As ρ is irreducible, we have either $W_0 = \mathcal{H}$ or $W_0 = \{0\}$.

If $W_0 = \mathcal{H}$ then ρ gives rise to a representation $\bar{\rho}$ of $H = \Gamma/A$. It is clear that $\bar{\rho}$ has no invariant vectors and thus for every $v \in \mathcal{H}$ with $\|v\| = 1$, we get

$$\frac{1}{|S| + |B|} \sum_{s \in S} \|\rho(s)v - v\|^2 = \frac{1}{|S| + |B|} \sum_{s \in S} \|\bar{\rho}(s)v - v\|^2 > \frac{\epsilon_H |S|}{|S| + |B|}.$$

We assume from now on that $W_0 = \{0\}$. The restriction of ρ to A decomposes into irreducible representations:

$$\rho = \oplus_{i=1}^n m_i \rho_i$$

where each ρ_i is one dimensional and the m_i are non negative integers. For each $i \in \{1, ..., n\}$, we denote the isotopic component $m_i \rho_i$ by χ_i and we write $X = \{\chi_i\}_{i=1}^n$. Then X is in one to one correspondence with a subset of the set of irreducible characters of A. By the irreducibility of ρ, H acts transitively on X (by conjugation). This action gives us a representation Φ of H on $\ell_2(X)$ (the vector space of all complex valued functions on X endowed with the standard inner product) defined by $(\Phi(h)F)(\chi) = F(\chi^h)$.

Now assume that for some ϵ which will be determined later, there exists a unit vector $v \in \mathcal{H}$, s.t

$$\frac{1}{|S| + |B|} \sum_{g \in \tilde{S} \cup B} \|\rho(g)v - v\|^2 \leq \epsilon.$$

Decompose v into its isotopic components: $v = \sum_{\chi \in X} v_\chi$. We define $F = F_v \in \ell_2(X)$ by $F(\chi) = \|v_\chi\|$. Note that F is a unit vector.

The following lemma is a general property of Kazhdan groups.

Lemma 2.1. *Let Γ be a group generated by a set S with a Kazhdan constant $\epsilon > 0$ with respect to S. Let $0 < \delta < \epsilon$ be given. Let $(\rho, \mathcal{H})$ be a unitary representation of Γ having a unit vector $v \in \mathcal{H}$, satisfying $\frac{1}{|S|} \sum_{s \in S} \|\rho(s)v - v\|^2 \leq \delta$. Then there exists an invariant vector v_0 satisfying $\|v - v_0\|^2 < \frac{\delta}{\epsilon}$.*

Proof. Decompose ρ into its trivial and non-trivial components, $\rho = \sigma_0 + \sigma_1$, and accordingly decompose $v = v_0 + v_1$.

Now σ_1 has no invariant vectors, therefore

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$$\delta \geq \frac{1}{|S|} \sum_{s \in S} \|\rho(s)v - v\|^2 = \frac{1}{|S|} \sum_{s \in S} \|\sigma_1(s)v_1 - v_1\|^2 > \epsilon \|v_1\|^2. \tag{1}$$

This implies that $\|v_1\|^2 < \frac{\delta}{\epsilon}$, hence we get $\|v - v_0\|^2 < \frac{\delta}{\epsilon}$ as required.

$\square$

Here is a technical lemma which we will use latter in the proof.

Lemma 2.2. *Let $\vec{a} = (a_1, ..., a_n) \in \mathbb{R}^n$ be a unit vector and $\vec{b} = (b_1, ..., b_n) \in \mathbb{R}^n$. Assume that there is some $K \in \mathbb{R}$ such that $|b_i| \leq K/\sqrt{n}$. Then*

$$\sum_{i=1}^n |a_i^2 - b_i^2| \leq \sum_{i=1}^n (a_i - b_i)^2 + 2K \sqrt{\sum_{i=1}^n (a_i - b_i)^2}.$$

Proof.

$$\sum_{i=1}^n \left(|a_i^2 - b_i^2| - (a_i - b_i)^2 \right) = \sum_{\substack{i=1 \\ a_i \geq b_i}}^n \left((a_i^2 - b_i^2) - (a_i - b_i)^2 \right) + \sum_{\substack{i=1 \\ b_i > a_i}}^n \left((b_i^2 - a_i^2) - (a_i - b_i)^2 \right) =$$

$$\sum_{\substack{i=1 \\ a_i \geq b_i}}^n 2b_i(a_i - b_i) + \sum_{\substack{i=1 \\ b_i > a_i}}^n 2a_i(b_i - a_i) \leq \frac{2K}{\sqrt{n}} \sum_{\substack{i=1 \\ a_i \geq b_i}}^n (a_i - b_i) + \frac{2K}{\sqrt{n}} \sum_{\substack{i=1 \\ b_i > a_i}}^n (b_i - a_i)$$

$$= \frac{2K}{\sqrt{n}} \sum_{i=1}^n |a_i - b_i| \leq 2K \sqrt{\sum_{i=1}^n (a_i - b_i)^2}.$$

The last inequality is by the Cauchy Schwarz inequality.

$\square$

Lemma 2.3. $\frac{1}{|S|} \sum_{s \in S} \|\Phi(s)F - F\|^2 \leq \frac{\epsilon(|S| + |B|)}{|S|}.$

Proof. From the definition of Φ, we have:

$$\frac{1}{|S| + |B|} \sum_{s \in S} \|\Phi(s)F - F\|^2 = \frac{1}{|S| + |B|} \sum_{s \in S} \sum_{\chi \in X} |F(\chi^s) - F(\chi)|^2 =$$

$$\frac{1}{|S| + |B|} \sum_{s \in S} \sum_{\chi \in X} |(\|v_{\chi^s}\| - \|v_\chi\|)|^2 = \frac{1}{|S| + |B|} \sum_{s \in S} \sum_{\chi \in X} |(\|v_{\chi^s}\| - \|\rho(s)v_\chi\|)|^2$$

$$\leq \frac{1}{|S| + |B|} \sum_{s \in S} \sum_{\chi \in X} \|\rho(s)v_\chi - v_{\chi^s}\|^2 \leq \epsilon.$$

$\square$

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Denote by $c = \frac{|B|}{|S|}$, now by Lemma 2.1 we obtain that there exists an invariant (i.e. constant) function G such that $\|F - G\|^2 < \frac{(1+c)\cdot\epsilon}{\epsilon_H}$. Thus

$$\|G\| = K < \|F\| + \sqrt{\frac{(1+c)\cdot\epsilon}{\epsilon_H}} = 1 + \sqrt{\frac{(1+c)\cdot\epsilon}{\epsilon_H}}$$

and so $G(\chi) = K/\sqrt{|X|}$. We choose $u \in \mathcal{H}$ such that for each $\chi \in X$,

$$G(\chi) = \|u_\chi\|.$$

Since $W_0 = \{0\}$, the following holds:

$$\epsilon_A < \frac{1}{|B||H|}\sum_{b\in B}\sum_{h\in H}\|\rho(b^h)v - v\|^2 = \frac{1}{|B||H|}\sum_{b\in B}\sum_{h\in H}\sum_{\chi\in X}\|\chi(b^h)v_\chi - v_\chi\|^2 =$$

$$\frac{1}{|B||H|}\sum_{b\in B}\sum_{h\in H}\sum_{\chi\in X}|\chi^h(b) - 1|^2\|v_\chi\|^2 = \frac{1}{|B||H|}\sum_{b\in B}\sum_{h\in H}\sum_{\chi\in X}|\chi(b) - 1|^2 F(\chi^{h^{-1}})^2.$$

Now,

$$\left|\frac{1}{|B||H|}\sum_{b\in B}\sum_{h\in H}\sum_{\chi\in X}|\chi(b) - 1|^2 F(\chi^{h^{-1}})^2 - \frac{1}{|B||H|}\sum_{b\in B}\sum_{h\in H}\sum_{\chi\in X}|\chi(b) - 1|^2 G(\chi)^2\right| \leq$$

$$\frac{1}{|B||H|}\sum_{b\in B}\sum_{h\in H}\sum_{\chi\in X}|\chi(b) - 1|^2 |F(\chi^{h^{-1}})^2 - G(\chi)^2| \leq$$

$$4 \cdot \frac{1}{|B||H|}\sum_{b\in B}\sum_{h\in H}\sum_{\chi\in X}|F(\chi^{h^{-1}})^2 - G(\chi)^2| = 4 \cdot \frac{1}{|B||H|}\sum_{b\in B}\sum_{h\in H}\sum_{\chi\in X}|F(\chi^h)^2 - G(\chi)^2|.$$

Now, substituting $a = F$ and $b = G$ in Lemma 2.2, we get:

$$4 \cdot \sum_{\chi\in X}|F(\chi^h)^2 - G(\chi)^2| \leq 4 \cdot \left(\sum_{\chi\in X}(F(\chi^h) - G(\chi))^2 + 2K\sqrt{\sum_{\chi\in X}(F(\chi^h) - G(\chi))^2}\right) =$$

$$4 \cdot \left(\sum_{\chi\in X}(F(\chi) - G(\chi))^2 + 2K\sqrt{\sum_{\chi\in X}(F(\chi) - G(\chi))^2}\right) \leq 4 \cdot \left(\|F - G\|^2 + 2K\sqrt{\|F - G\|^2}\right)$$

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$$< 4 \cdot \left(\frac{(1+c) \cdot \epsilon}{\epsilon_H} + 2(1 + \sqrt{\frac{(1+c) \cdot \epsilon}{\epsilon_H}}) \sqrt{\frac{(1+c) \cdot \epsilon}{\epsilon_H}} \right) = 12 \cdot \frac{(1+c) \cdot \epsilon}{\epsilon_H} + 8 \cdot \sqrt{\frac{(1+c) \cdot \epsilon}{\epsilon_H}}$$

Now

$$\epsilon_A < \frac{1}{|B||H|} \sum_{b \in B} \sum_{h \in H} \|\rho(b^h)v - v\|^2 = \frac{1}{|B||H|} \sum_{b \in B} \sum_{h \in H} \sum_{\chi \in X} |\chi(b) - 1|^2 \|v_{\chi^{h-1}}\|^2 \leq$$

$$12 \cdot \frac{(1+c) \cdot \epsilon}{\epsilon_H} + 8\sqrt{\cdot \frac{(1+c)^2 \cdot \epsilon}{\epsilon_H}} + \frac{1}{|B||H|} \sum_{b \in B} \sum_{h \in H} \sum_{\chi \in X} |\chi(b) - 1|^2 G(\chi)^2 \leq$$

$$12 \cdot \frac{(1+c) \cdot \epsilon}{\epsilon_H} + 8\sqrt{\cdot \frac{(1+c) \cdot \epsilon}{\epsilon_H}} + 12 \cdot \frac{(1+c) \cdot \epsilon}{\epsilon_H} + 8\sqrt{\cdot \frac{(1+c) \cdot \epsilon}{\epsilon_H}} + \frac{1}{|B||H|} \sum_{b \in B} \sum_{h \in H} \sum_{\chi \in X} |\chi(b)-1|^2 \|v_\chi\|^2 =$$

$$24 \cdot \frac{(1+c) \cdot \epsilon}{\epsilon_H} + 16\sqrt{\cdot \frac{(1+c) \cdot \epsilon}{\epsilon_H}} + \frac{1}{|B||H|} \sum_{b \in B} \sum_{h \in H} \sum_{\chi \in X} |\chi(b) - 1|^2 \|v_\chi\|^2 =$$

$$24 \cdot \frac{(1+c) \cdot \epsilon}{\epsilon_H} + 16 \cdot \frac{(1+c) \cdot \epsilon}{\epsilon_H} + \frac{1}{|B|} \sum_{b \in B} \sum_{\chi \in X} |\chi(b) - 1|^2 \|v_\chi\|^2 <$$

$$24 \cdot \frac{(1+c) \cdot \epsilon}{\epsilon_H} + 16\sqrt{\cdot \frac{(1+c) \cdot \epsilon}{\epsilon_H}} + (1 + \frac{1}{c})\epsilon.$$

If we choose

$$\epsilon \leq \frac{\epsilon_H \epsilon_A}{512 \cdot (1 + c + \frac{1}{c})},$$

we get a contradiction (the last inequality follows from the fact that $\epsilon, \epsilon_H \leq 1$) and this finishes the proof of Theorem 1.6.

$\square$

3 Applications

3.1 Solvable Groups

Lubotzky and Weiss ([LW], Corollary 3.3) proved that Kazhdan constants of an infinite family of solvable groups with bounded derived length, with respect to any bounded generating set

are going to zero. The general way to give a positive Kazhdan constant for an infinite family of solvable groups is by using Alon-Roichman result which require a generating set of logarithmic size of the group.

The first improvement for some solvable groups is due to Meshulam and Wigderson [MW] (based on [RVW, ALW]). They constructs a family of solvable groups of the form $G_{i+1} = G_i \ltimes \mathbb{F}_{p_i}[G_i]$ where G_i is a finite group, p_i is a prime and G_{i+1} is the natural semi-direct product.

Theorem 3.1. *([MW], Theorem 1.7) There exists a group G_1 and a sequence of primes p_i, such that for any $0 < \epsilon < 1$ we have*

$$g_\epsilon(G_n) \leq \log^{(n-\log^* n)} |G_n|,$$

where $\log^k$ denotes the k times iterated logarithm function.

We would like to mention a related work by Rozenman, Shalev and Wigderson [RSW] (which is also based on [RVW, ALW]). In this work they gave an iterative construction of infinite family of finite (non-solvable) groups $\{G_n\}_{n=1}^\infty$ and they showed that for any $n \in \mathbb{N}$, $g_\epsilon(G_n) = O(1)$ for some $\epsilon > 0$.

Influenced by Meshulam and Wigderson methods, we will give in this section a construction of a family of meta-abelian groups Γ_p (split and non-split), with a substantial improvement for $g_\epsilon(\Gamma_p)$ (compared to Alon-Roichman).

3.2 Metabelian Groups

3.2.1 Group Cohomology and Non-Splitting Extension

Definition 3.2. If $\pi : \Gamma \to H$ is surjective homomorphism of groups, then a *lifting* of $x \in H$ is an element $l(x) \in \Gamma$ with $\pi(l(x)) = x$.

Let A be a group, H a group with a homomorphism

$$\theta : H \to Aut(A).$$

Let $H^2(H, A, \theta)$ be the second cohomology group. As is well known (see [Rot], ch. 7), $H^2(H, A, \theta)$ parameterizes groups extension of A by H with respect to θ.

Let p be a prime, $1 \leq n \in \mathbb{N}$ and let C_{p^n} be the cyclic group of order p^n. In the next lemma we will prove the existence of non-splitting extension of a p-group by some cyclic groups. Next, by using this lemma we will construct a families of metabelian groups with non-trivial second cohomology.

Lemma 3.3. *Let p be a prime, $2 \leq n \in \mathbb{N}$ and F a non-trivial finite p-group. Assume that the cyclic group C_{p^n} has non-trivial action θ on F and for some $k < n$, $p^k \cdot C_{p^n}$ acts trivially on F, and F has exponent dividing p^{n-k} (i.e. $x^{p^{n-k}} = 1$ for all $x \in F$). Then*

$$H^2(C_{p^n}, F, \theta) \neq 0.$$

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Proof. Let $\Gamma = C_{p^n} \ltimes F$ which is a finite p-group, F is a non-trivial normal subgroup so has non-trivial intersection with the center. So there exists a subgroup Z of F, cyclic of order p, which is central in $C_{p^n} \ltimes F$. Now the cyclic group $C_{p^{n+1}}$ has a unique subgroup of order p which we denote by Z'. Let $\tilde{\Gamma} = C_{p^{n+1}} \ltimes F$. Consider a "diagonal subgroup" K in $Z \times Z'$. Then it is easy to check that F is a normal subgroup of $\tilde{\Gamma}/K$. So $\tilde{\Gamma}$ is an extension of F by C_{p^n}, We claim that the generator of C_{p^n} cannot be lifted to an element of order p^n. Indeed, if (c, f) is a lifting, then taking the power p^k we get an element (c^{p^k}, f') and c^{p^k} centralizes f', so taking the power p^{n-k} of the latter (i.e. the power p^n of the initial element) and using the exponent, we obtain $(c^{p^n}, 1)$; this element of $C_{p^{n+1}} \ltimes F$ remains non-trivial after taking the quotient by K.

$\square$

Let $\mathbb{F}_p$ be the field of order p and $\mathbb{F}_p[C_p]$ the group algebra. Every element $f \in \mathbb{F}_p[C_p]$ can be represented as $f = \sum_{x \in C_p} f(x)x$.

Let $\theta : C_{p^n} \to Aut(\mathbb{F}_p[C_p])$ be the left translation (mod p) homomorphism. As a consequence of last lemma (take $k = 1$) we get the following result:

Lemma 3.4. *For every $n \geq 2$*

$$H^2(C_{p^n}, \mathbb{F}_p[C_p], \theta) \neq 0,$$

hence, there is a non-split extension of $A = \mathbb{F}_p[C_p]$ by $H = C_{p^n}$.

3.2.2 Proof of Corollary 1.7:

We begin by proving that for any $0 < \epsilon < 1$, there exist a subset $B \subseteq \mathbb{F}_p[C_p]$ such that the average Kazhdan constant of $\mathbb{F}_p[C_p]$ with respect to $\tilde{B} = B^{C_p}$ is greater than ϵ.

Given $f, g \in \mathbb{F}_p[C_p]$, define the product $f \cdot g = \sum_{x \in C_p} f(x)g(x) \in \mathbb{F}_p$. Let $e_p(\alpha) = \exp(\frac{2\pi\alpha i}{p})$.

A multiset $\tilde{B} \subset \mathbb{F}_p[C_p]$ is called δ-balanced if for all $0 \neq f \in \mathbb{F}_p[C_p]$

$$|\sum_{h \in \tilde{B}} e_p(f \cdot h)| \leq (1 - \delta)|\tilde{B}|.$$

If $\tilde{B}$ is δ-balanced then

$$1 - \left(\frac{1}{|\tilde{B} \cup \tilde{B}^{-1}|}|\sum_{h \in \tilde{B} \cup \tilde{B}^{-1}} e_p(f \cdot h)|\right) = 1 - \frac{1}{2|\tilde{B}|}|\sum_{h \in \tilde{B}} e_p(f \cdot h) + \sum_{h \in \tilde{B}^{-1}} e_p(f \cdot h)| \geq$$

$$1 - \frac{1}{2|\tilde{B}|}\left(|\sum_{h \in \tilde{B}} e_p(f \cdot h)| + |\sum_{h \in \tilde{B}^{-1}} e_p(f \cdot h)|\right) = 1 - \frac{1}{|\tilde{B}|}|\sum_{h \in \tilde{B}} e_p(f \cdot h)| \geq \delta.$$

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Since $\mathbb{F}_p[C_p]$ is an abelian group, it is easy to check that the spectral gap of $\mathbb{F}_p[C_p]$ with respect to $\tilde{B} \cup \tilde{B}^{-1}$ is:

$$\min_{\chi \neq 1} \left| \frac{1}{|\tilde{B} \cup \tilde{B}^{-1}|} \sum_{h \in \tilde{B} \cup \tilde{B}^{-1}} \chi(h) - 1 \right|.$$

So, from the discussion in subsection 1.2, we obtain that if $\tilde{B}$ is δ-balanced, then the average Kazhdan constant of A with respect to $\tilde{B} \cup \tilde{B}^{-1}$ is at least 2δ.

For $f \in \mathbb{F}_p[C_p]$, $s \in \mathbb{N}$ and $\delta > 0$ let

$$B_\delta(f) = \left\{ (h_1, ..., h_s) \in \mathbb{F}_p[C_p]^s : \left| \frac{1}{sp} \sum_{i=1}^{s} \sum_{\sigma \in C_p} e_p(\sigma h_i \cdot f) \right| > 1 - \delta \right\}.$$

For $f \in \mathbb{F}_p[C_p]$, define a linear map $T_f : \mathbb{F}_p[C_p] \to \mathbb{F}_p[C_p]$ by $T_f(h) = hf$. Denote by $C_p f$ the orbit of f under the action of C_p. It is clear that

$$\dim(Span\{C_p f\}) = \mathrm{rank} T_f.$$

Let

$$V_r(\mathbb{F}_p) = \{f \in \mathbb{F}_p[C_p] : \mathrm{rank} T_f = r\}.$$

Let C_{p^n} be the cyclic group of order p^n for some $n \in \mathbb{N}$. We defined above the following representation of C_{p^n}:

$$\theta : C_{p^n} \to Aut(\mathbb{F}_p[C_p]).$$

Let a be a generator of C_{p^n}, set $\alpha = \theta(a)$, then α satisfy $\alpha^{p^n} = 1$. Thus α is a root of the polynomial $x^{p^n} - 1 = (x-1)^{p^n}$. So the minimum polynomial of α is $(x-1)^k$ for some $k \leq p^n$. Hence, 1 is the only eigenvalue of α, so the trivial representation (i.e. this is the subspace of constant vectors) is the only irreducible representation of C_{p^n}.

Now it is immediate to check that $\mathbb{F}_p[C_p]$ is an indecomposable representation of dimension p. Furthermore, consider the Jordan canonical form of α, we see that a subspace of $\mathbb{F}_p[C_p]$ is indecomposable if and only if it correspond to a Jordan block which may be of size $1 \leq s \leq p$. Therefore, if one represent α in it Jordan canonical from, then we get that for every $1 \leq k \leq p$ the number of elements in $\mathbb{F}_p[C_p]$ of rank k, is $(p-1)p^{k-1}$.

Meshulam and Wigderson (see Proposition 3.2 in [MW]) proved the following:

Proposition 3.5. *If* $\mathrm{rank} T_f = r$ *then*

$$Prob(B_\delta(f)) \leq 8 \exp\left(\frac{-(1-2\delta)^2 rs}{4} \right).$$

The following proposition is a variant of Theorem 1.2 in [MW].

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Proposition 3.6. *For any $0 < \delta < \frac{1}{2}$ there exist $s = O\left(\frac{1}{(1-2\delta)^2} \cdot \ln p\right)$ elements $h_1, ..., h_s \in \mathbb{F}_p[C_p]$ such that the multiset $\tilde{B} = \bigcup_{i=1}^s C_p h_i \subset \mathbb{F}_p[C_p]$ is δ-balanced and the average Kazhdan constant of $\mathbb{F}_p[C_p]$ with respect to $\tilde{B} \cup \tilde{B}^{-1}$ is 2δ.*

Proof.

$$Pr\left(\bigcup_{0 \neq f \in \mathbb{F}_p[C_p]} B_\delta(f)\right) \leq \sum_{0 \neq f \in \mathbb{F}_p[C_p]} B_\delta(f) \leq$$

$$8 \sum_{r \geq 1} |V_r(\mathbb{F}_p)| \exp\left(\frac{-(1-2\delta)^2 rs}{4}\right) \leq$$

$$8 \sum_{r=1}^p p^r \exp\left(\frac{-(1-2\delta)^2 rs}{4}\right) = 8 \sum_{r=1}^p (p^4 \exp(-(1-2\delta)^2 s)^{\frac{r}{4}} =$$

$$8[p^4 \exp(-(1-2\delta)^2 s)]^{\frac{1}{4}} \frac{1 - [p^4 \exp(-(1-2\delta)^2 s]^{\frac{p}{4}}}{1 - [p^4 \exp(-(1-2\delta)^2 s]^{\frac{1}{4}}} < 1$$

For any $c > 1$ and for p large enough, if one choose $s = \frac{4}{(1-2\delta)^2}(c \ln p)$, then the probability that $h_1, ..., h_s$ is not δ-balanced is strictly less than 1 and the result follow. $\square$

The last proposition still holds if we replace the group C_p by C_{p^n} with the action by left translation mod p on $\mathbb{F}_p[C_p]$. This follow easily from the fact that for every $f \in \mathbb{F}_p[C_p]$, every element in the multiset $C_p f$ appears p^{n-1} times in the multiset $C_{p^n} f$.

Now we are ready to complete the proof of Corollary 1.7. By [AR] for any $0 < \epsilon_1 < 1$, there is a generating set S for C_{p^n} of size $O_{\epsilon_1}(\log |C_{p^n}|)$ such that the Kazhdan constant of C_{p^n} with respect to S is bigger than ϵ_1. Let Γ be any extension of $\mathbb{F}_p[C_p]$ by C_{p^n} with respect to to the action of left translation. Let $\tilde{S}$ be any lifting of S to Γ, from Proposition 3.6 (and the comment afterward), we get that for any $0 < \epsilon_2 < 1$ there exist a set $B \subset \mathbb{F}_p[C_p]$ such that the Kazhdan constant of $\mathbb{F}_p[C_p]$ with respect to $\tilde{B} = B^{C_{p^n}}$ is greater then ϵ_2. Now from Theorem 1.6 the proof is complete. $\square$

3.3 p-Groups

Fix a prime p, for any group Γ, define Γ^p to be the subgroup generated by $\{g^p : g \in \Gamma\}$. The lower p-series (also called the lower central p-series or the lower exponent-p series) of Γ is the descending series

$$\Gamma = \varphi_1(\Gamma) \geq \varphi_2(\Gamma) \geq \cdots \geq \varphi_n(\Gamma) \geq \cdots,$$

defined by

$$\varphi_{i+1}(\Gamma) = \varphi_i(\Gamma)^p [\varphi_i(\Gamma), \Gamma]$$

for $i \geq 1$. The group Γ is said to have class n, if $\varphi_n(\Gamma)$ is the last non-identity element of the lower p-series.

This subsection uses results and properties of the lower p-series. Proofs of the results can be found in Huppert and Blackburn [HB] (ch. VIII) and in Blackburn, et el. [BNV] (ch. 4). On can check the following:

Proposition 3.7. *For all positive integer i, $\varphi_{i+1}(\Gamma)$ is the smallest normal subgroup of Γ lying in $\varphi_i(\Gamma)$, such that $\varphi_i(\Gamma)/\varphi_{i+1}(\Gamma)$ is an elementary abelian p group and is central in $\Gamma/\varphi_{i+1}(\Gamma)$.*

Let F_k be the free group on k generators. Let $\Gamma_{k,n} = F_k/\varphi_{n+1}(F_k)$. This group is a p-group, called the relatively free group on k generators in the variety of p-groups with a lower-p series of class n (for an introduction to variety of a groups see ch. 1 in [Ne]). The dimensions of $\varphi_i(F_k)/\varphi_{i+1}(F_k)$ can be calculated explicitly (see [HB]), in particular for the case of $\Gamma_{k,2} = F_k/\varphi_3(F_k)$, we have the following lemma:

Lemma 3.8. *([BNV], Lemma 4.2) The Frattini subgroup $\Phi(\Gamma_{k,2})$ of $\Gamma_{k,2}$ is of order $p^{\frac{1}{2}k(k+1)}$ and index p^k.*

As a consequence of the last lemma, one can show that the order of the commutator subgroup $\Gamma'_{k,2}$ is $p^{\frac{1}{2}k(k-1)}$.

3.3.1 Proof of Corollary 1.9:

Let $\Gamma_{k,2}$ be the relatively free group in the variety of p-groups with a lower-p series of class. The Frattini subgroup of $\Gamma_{k,2}$ is

$$\Phi(\Gamma_{k,2}) = \Gamma_{k,2}{}^p[\Gamma_{k,2}, \Gamma_{k,2}] = \varphi_2(\Gamma_{k,2}).$$

Hence if $\tilde{S} \subset \Gamma_{k,2}$ is such that its image in $\Gamma_{k,2}/\varphi_2(\Gamma_{k,2})$ generates $\Gamma_{k,2}/\varphi_2(\Gamma_{k,2})$, then $\tilde{S}$ generates $\Gamma_{k,2}$.

From Proposition 3.7, $\varphi_2(\Gamma_{k,2})$ is central in $\Gamma_{k,2}$, so $\Gamma_{k,2}$ is a central extension of $\Gamma'_{k,2} = [\Gamma_{k,2}, \Gamma_{k,2}]$ by $\Gamma_{k,2}/\Gamma'_{k,2}$.

By [AR] for any $0 < \epsilon < 1$ there is a generating set S for $\Gamma_{k,2}/\Gamma'_{k,2}$ of size $O_\epsilon(\log |\Gamma_{k,2}/\Gamma'_{k,2}|) = O_\epsilon(\log |p^{2k}|)$ such that the average Kazhdan constant of $\Gamma_{k,2}/\Gamma'_{k,2}$ with respect to S is bigger than ϵ. Let $\tilde{S}$ be any lifting of S to $\Gamma_{k,2}$. We claim that the average Kazhdan constant of $\Gamma_{k,2}$ with respect to $\tilde{S}$ is bigger than $\epsilon/4$. Indeed, one dimensional representations are trivial on $\Gamma'_{k,2}$ (since it is the commutator subgroup) and hence factor through $\Gamma_{k,2}/\Gamma'_{k,2}$, and for higher dimensional representations we use theorem 1.8. We have shown that $\Gamma_{k,2}$ has a generating set of size

$$O_\epsilon(\log |\Gamma_{k,2}/\Gamma'_{k,2}|) = O_\epsilon(\sqrt{\log |\Gamma_{k,2}|}).$$

One can repeat the above methods and improve $g_\epsilon(\Gamma_{k,n})$ for the relatively free group on k generators in the variety of p-groups with a lower-p series of class $n > 2$. $\qquad\square$

3.4 Kazhdan Constants for the Tame Automorphism Groups of Free Nilpotent Groups

Let F_k be the free group on k generators $x_1, ..., x_k$. For any $1 \leq i \neq j \leq k$, let $R_{i,j}^\pm, L_{i,j}^\pm$ be the following automorphisms of F_k:

$$R_{i,j}^\pm(x_i) = x_i x_j^{\pm 1} \text{ and } R_{i,j}^\pm(x_l) = x_l \text{ if } l \neq i.$$

$$L_{i,j}^\pm(x_i) = x_j^{\pm 1} x_i \text{ and } L_{i,j}^\pm(x_l) = x_l \text{ if } l \neq i.$$

These automorphisms are called Nielsen moves. Let S be the set of Nielsen moves of the free group on k generators, it is easy to check that $|S| = 4k(k-1)$.

Definition 3.9. Let F_k be the free group on k generators and $W = \gamma_{c+1}(F_k)$ be the $(c+1)$-th term of the lower central series of F_k. Then F_k/W is called the free nilpotent group of class c and will be denoted by $F_k(c)$.

Let U be a characteristic subgroup of F_k, then it is clear that every automorphism of F_k induces an automorphism on F_k/U. This fact bring us to the following definition.

Definition 3.10. An automorphism of the free nilpotent group $F_k(c)$ is said to be *tame* if it is in the image of the natural projection map $Aut(F_k) \to Aut(F_k(c))$. The subgroup of all the tame automorphisms is denoted by $A_k(c)$.

The group $A_k(c)$ has the following short exact sequence:

$$1 \to K_1 \to A_k(c) \to SL_k(\mathbb{Z}) \to 1$$

where K_1 is a nilpotent group of class $c - 1$ (see [And] for details).

Lubotzky and Pak showed in [LP] that for $k \geq 3$, the group $A_k(c)$ is a lattice in some Lie group which has Kazhdan property (T), hence it has property (T). Their approach does not give an estimate for the Kazhdan constant. Mallahi [MAK] following Burger, Shalom and Kassabov (see [Bur, Sh1, Kas1]), used the spectral measure on the dual group $\hat{\mathbb{Z}}^m$ correspond to $SL_k(\mathbb{Z})$ and prove that the group $A_k(c)$ has relative property (T) with respect to the center of K_1. Before we state Mallahi results in a more general setting (which follows easily from Mallahi's proof), we give some notation. Let $k, c \in \mathbb{N}$, we define

$$\delta(k,c) = \frac{2\sqrt{c}}{\sqrt{(84\sqrt{k} + 1920)(4k)^c}}.$$

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Theorem 3.11. *(Theorem 2.5.6 in [MAK]) Let $k \geq 3$ be an integer, $A_k(c)$ be the group of tame automorphisms of the free nilpotent group of class c and S the set of Nielsen transformations. Let A be the center of $A_k(c)$ and let Z be the center of K_1/A. Let $(\rho, \mathcal{H})$ be a unitary representation of $A_k(c)/A$ with $(S, \delta(k, c))$-invariant vector. Then $\mathcal{H}$ contains a non-zero Z-invariant vector.*

From the last theorem, Mallahi deduce the following:

Theorem 3.12. *(Theorem 2.5.7 in [MAK]) Let $k \geq 3$ be an integer, $A_k(c)$ be the group of tame automorphisms of the free nilpotent group of class c and S the set of Nielsen transformations. If $c < 2k+1$, then the Kazhdan constant $\epsilon(k, c)$ of $A_k(c)$ with respect to S is greater than or equal to*

$$\frac{\sqrt{c}}{\sqrt{(84\sqrt{k} + 1920)(4k)^c}}.$$

The limitation for $c < 2k + 1$ is based on the following fact due to Formanek (Theorem 6 in [For]):

Theorem 3.13. *Let $F_k(C)$ be the free nilpotent group of rank k and class c where $k, c \geq 2$. Then the automorphism group has a non-trivial center iff $c = 2kl + 1$ for $l \geq 1$.*

The goal of this section is to extend Mallahi result and calculate Kazhdan constant of $A_k(c)$ for any $k \geq 3$ and $c \geq 1$. Using the same notations as above we prove:

Theorem 3.14. *Let $k \geq 3$ be an integer, $A_k(c)$ be the group of the tame automorphisms of the free nilpotent group of class c and S the set of Nielsen transformations, then the Kazhdan constant $\epsilon(k, c)$ of $A_k(c)$ with respect to S satisfies:*

$$\epsilon(k,c) \geq \begin{cases} \frac{\sqrt{c}}{\sqrt{(84\sqrt{k}+1920)4^{c+3}k^{c+4}}} & \text{if } c = 2kl+1 \text{ for some } l \in \mathbb{N}; \\ \frac{\sqrt{c-1}}{\sqrt{(84\sqrt{k}+1920)4^{c+3}k^{c+3}}} & \text{if } c = 2kl+2 \text{ for some } l \in \mathbb{N}; \\ \frac{\sqrt{c-2}}{\sqrt{(84\sqrt{k}+1920)4^{c+3}k^{c+2}}} & \text{if } c = 2kl+3 \text{ for some } l \in \mathbb{N}; \\ \frac{\sqrt{c-3}}{\sqrt{(84\sqrt{k}+1920)4^{c+3}k^{c+1}}} & \text{if } c = 2kl+4 \text{ for some } l \in \mathbb{N} \\ \frac{\sqrt{c-4}}{\sqrt{(84\sqrt{k}+1920)4^{c+3}k^{c}}} & \text{otherwise.} \end{cases}$$

Proof. For the case $c \leq 2k$, by Theorem 3.12 we get that the Kazhdan constant of $A_k(c)$ with respect to S is

$$\epsilon(k,c) \geq \frac{\sqrt{c}}{\sqrt{(84\sqrt{k} + 1920)(4k)^c}}.$$

For the case $c = 2k + 1$, denote by A the center of $A_k(c)$, by Theorem 3.13, A is not trivial. Hence we have the following exact sequence:

$$1 \to A \to A_k(c) \to A_k(c)/A \to 1,$$

and respectively

$$1 \to K_1/A \to A_k(c)/A \to SL_k(\mathbb{Z}) \to 1.$$

Lemma 3.15. *Let $c = 2k + 1$ then the Kazhdan constant of $A_k(c)/A$ is greater than or equal to $\frac{\delta(k,c)}{2}$.*

Proof. Let $(\rho, , \mathcal{H})$ be a unitary representation of $A_k(c)/A$ and assume that there exist a unit vector $v \in \mathcal{H}$ which is $(\frac{\delta(k,c)}{2}, S)$-invariant. Let Z be the center of K_1/A, put

$$\mathcal{H}_0 = \{v \in \mathcal{H} : \rho(z)v = v, \forall z \in Z\}$$

and $\mathcal{H}_1 = \mathcal{H}_0^\perp$ (the orthogonal complement). Write $v = v_0 + v_1$ where $v_i \in \mathcal{H}_i$, by Theorem 3.11 we get that $\mathcal{H}_0 \neq 0$.

The subspace $\mathcal{H}_1$ does not have a Z-invariant vector, and therefore it does not have a $(\delta(k,c), S)$-invariant vector. This implies that $\delta(k,c)\|v_1\| < \max_{s \in S} \|\rho(s)v_1 - v_1\| \leq \frac{\delta(k,c)}{2}$, and we deduce that $\|v_1\| < \frac{1}{2}$.

So $\|v_0\| \geq \frac{1}{2}$ and $\mathcal{H}_0$ is $A_k(c)/A$-invariant because Z is a normal subgroup. Furthermore, $Z = \gamma_c(K_1/A)$, therefore it rise a unitary representation $\overline{\rho}$ of

$$(A_k(c)/A)/Z = A_k(c-1)$$

on $\mathcal{H}_0$.

Let $\tilde{S}$ be the projection of the generators S to $A_k(c-1)$, then it is clear that $\tilde{S}$ is the set of Nielsen transformations for $A_k(c-1)$. Now, for any $s \in S$ we have

$$\|\rho(s)v - v\|^2 = \|\overline{\rho}(s)v_0 - v_0\|^2 + \|\rho(s)v_1 - v_1\|^2 \leq \frac{\delta(k,c)^2}{4}$$

and therefore

$$\|\overline{\rho}(s)v_0 - v_0\|^2 \leq \frac{\delta(k,c)^2}{4} \leq \delta(k,c)^2 \|v_o\|^2 \leq \frac{\delta(k,c-1)^2}{4} \|v_0\|^2.$$

Consequently, the vector $v_0 \in \mathcal{H}_0$ is $(\frac{\delta(k,c-1)}{2}, \tilde{S})$-invariant . By Theorem 3.12 we obtain that $\mathcal{H}_0$ contains a non-zero invariant vector of $A_k(c-1)$ and this vector is also an invariant vector of $A_k(c)/A$. $\square$

The group $A_k(c)$ does not have a non-trivial representation of dimension one, because $A_k(c) = [A_k(c), A_k(c)]$ (see [Ge]) and $|S| = 4k(k-1)$, therefore from Theorem 1.8 we obtain that the Kazhdan constant $\epsilon(k, c)$ of $A_k(c)$ for $c = 2k + 1$ with respect to S is greater than or equal to

$$\frac{\delta(k, c)}{2|S|} \geq \frac{\sqrt{c}}{\sqrt{(84\sqrt{k} + 1920)4^{c+3}k^{c+4}}}.$$

For $c > 2k + 1$, we continue by induction on c and repeating the same arguments as in Lemma 3.15.

□

4   Acknowledgments.

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